Nef and big divisors on toric 3-folds with nef anti-canonical divisors[*]

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Abstract

We show that an ample line bundle on a nonsingular complete toric 3-fold with nef anti-canonical divisor is normally generated. As a consequence of our proof, we see that an ample line bundle whose adjoint bundle has global sections on a Gorenstein toric Fano 3-fold is normally generated.

Introduction

We call an invertible sheaf on an algebraic variety a line bundle. A line bundle L on an algebraic variety is called *normally generated* (by Mumford[14]) if the multiplication map of global sections $\Gamma(L)^{\otimes l} \to \Gamma(L^{\otimes l})$ is surjective for all $l \geq 1$. We are interested in normal generation of ample line bundles on a toric variety. If an ample line bundle L on a normal algebraic variety X is normally generated, then we see that it is very ample and that the graded ring $\bigoplus_{l \geq 0} \Gamma(X, L^{\otimes l})$ is generated by elements of degree one and is a normal ring. It is known that an ample line bundle on a nonsingular toric variety is always very ample (see [18, Corollary 2.15]). We may ask whether any ample line bundle be normally generated.

In general, for an ample line bundle L on a (possibly singular) toric variety of dimension n, we see that

$$\Gamma(L^{\otimes l}) \otimes \Gamma(L) \longrightarrow \Gamma(L^{\otimes(l+1)}) \tag{1}$$

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is surjective for $l \geq n - 1$ (see [1], [16] or [17]). When $n \leq 2$, hence, we see that all ample line bundles are normally generated (see [9]). We also have examples of ample and not normally generated line bundles for $n \geq 3$.

We know that the anti-canonical line bundle on a nonsingular toric Fano variety of dimension n is normally generated if $n \leq 7$ (see [7]). Ogata[20] shows that an ample line bundle L on a nonsingular toric 3-fold X with $h^0(L + 2K_X) = 0$ is normally generated.

In this paper we restrict X to be a nonsingular toric 3-fold with nef anti-canonical divisor.

Theorem 1 *Let X be a nonsingular toric variety of dimension three with nef $-K_X$. If a nef and big line bundle L on X satisties that $2L + K_X$ is nef and $h^0(L + K_X) \neq 0$, then L is normally generated.*

Combining this with the result of [20], we obtain the following theorem.

Theorem 2 *Ample line bundles on a nonsingular toric 3-fold with nef ant-canonical divisor are normally generated.*

Since a Gorenstein toric Fano 3-fold admits a crepant resolution, Theorem 1 implies the following theorem.

Theorem 3 *Let Y be a Gorenstein toric Fano variety of dimension three. If an ample line bundle L on Y satisfies that $h^0(L + K_Y) \neq 0$, then L is normally generated.*

In our proof we do not use classifications of Fano polytopes. There are 4,319 Gorenstein toric Fano 3-folds (cf. [11]).

We note that there is an ample but not normally generated line bundle L on a Gorenstein toric Fano 3-fold Y with $h^0(L + K_Y) = 0$.

1 Line bundles on toric varieties

In this section we recall the fact about toric varieties and line bundles on them from Oda's book[18] or Fulton's book[5].

Let N be a free $\mathbb{Z}$-module of rank n and $M := \text{Hom}(N, \mathbb{Z})$ its dual with the pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. By scalar extension to $\mathbb{R}$, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. We also have the pairing of $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ by scalar extension, which is denoted by the same symbol $\langle \cdot, \cdot \rangle$.

The group ring $\mathbb{C}[M]$ defines an algebraic torus $T_N := \text{Spec}\,\mathbb{C}[M] \cong (\mathbb{C}^*)^n$ of dimension n. Then the character group $\text{Hom}_{\text{gr}}(T_N, \mathbb{C}^*)$ of the algebraic

torus T_N coincides with M. For $m \in M$ we denote the corresponding character by $e(m) : T_N \to \mathbb{C}^*$.

Let Δ be a finite complete fan of N. A convex cone $\sigma \in \Delta$ defines an affine variety $U_\sigma := \mathrm{Spec}\, \mathbb{C}[M \cap \sigma^\vee]$. Here $\sigma^\vee := \{y \in M_\mathbb{R}; \langle y, x \rangle \geq 0 \quad \text{for all} \quad x \in \sigma\}$ is the dual cone of σ. Then we obtain a normal algebraic variety $X(\Delta) := \bigcup_{\sigma \in \Delta} U_\sigma$, which is called a *toric* variety. We note that $U_{\{0\}} \cong T_N$ is a unique dense T_N-orbit in $X(\Delta)$. Set $\Delta(i) := \{\sigma \in \Delta; \dim \sigma = i\}$. Then an element $\sigma \in \Delta(i)$ corresponds to a T_N-invariant subvariety $V(\sigma)$ of dimension $n-i$. In particular, $\Delta(1)$ corresponds to the set of all irreducible T_N-invariant divisors on $X(\Delta)$.

Let $\Delta(1) = \{\rho_1, \ldots, \rho_s\}$ and v_i the generator of the semi-group $\rho_i \cap N$. We simply write as $X = X(\Delta)$ and $D_i := V(\rho_i)$ for $i = 1, \ldots, s$. For a T_N-invariant line bundle L there exists a T_N-invariant divisor $D = \sum_i a_i D_i$ satisfying $L \cong \mathcal{O}_X(D)$. For a T_N-invariant Cartier divisor D we define a rational convex polytope $P_D \subset M_\mathbb{R}$ as

$$P_D := \{y \in M_\mathbb{R}; \langle y, v_i \rangle \geq -a_i \quad \text{for} \quad i = 1, \ldots, s\}. \tag{2}$$

By definition we note that $P_{lD} = lP_D$ for any positive integer l. Moreover, for another T_N-invariant Cartier divisor E we have $P_{D+E} \supset P_D + P_E$. Here $P_D + P_E := \{x + y \in M_\mathbb{R}; x \in P_D \quad \text{and} \quad y \in P_E\}$ is the Minkowski sum of P_D and P_E. By using this polytope, we can describe the space of global sections (see [18, Section 2.2], or [5, Section 3.5])

$$\Gamma(X, \mathcal{O}_X(D)) \cong \bigoplus_{m \in P_D \cap M} \mathbb{C}e(m). \tag{3}$$

If $\mathcal{O}_X(D)$ is generated by global sections, then all vertices of P_D are lattice points, that is, P_D is the convex hull of a finite subset of M. Conversely, if for all $\sigma \in \Delta$ there exist $u(\sigma) \in M$ with

$$\langle u(\sigma), v_i \rangle = -a_i \quad \text{for} \quad v_i \in \sigma \tag{4}$$

and if P_D is the convex hull of $\{u(\sigma); \sigma \in \Delta\}$, then $\mathcal{O}_X(D)$ is generated by global sections (see [18, Theorem 2.7], or [5, Section 3.4]).

If $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ are generated by global sections, then we have $P_{D+E} = P_D + P_E$. In this case, from the equality (3) we see that the surjectivity of the multiplication map of global sections

$$\Gamma(X, \mathcal{O}_X(D)) \otimes \Gamma(X, \mathcal{O}_X(E)) \longrightarrow \Gamma(X, \mathcal{O}_X(D+E)) \tag{5}$$

is equivalent to the equality

$$P_D \cap M + P_E \cap M = (P_D + P_E) \cap M. \tag{6}$$

We also knows [12] that if $\mathcal{O}_X(D)$ is generated by global sections, then there exists an equivariant surjective morphism $\pi : X \to Y$ to a toric variety Y and an ample line bundle A on Y with $\mathcal{O}_X(D) \cong \pi^* A$. From [15, Theorem 3.1] we know that $\mathcal{O}_X(D)$ is generated by global sections if and only if D is *nef*.

If X is Gorenstein, then $-K_X = \sum_i D_i$ is a Cartier divisor. By definition P_{-K_X} is a rational polytope of dimension n since the polytope is the intersection of half-spaces containing the origin as their interiors. This implies that $-K_X$ is *big*.

Now we introduce a criterion of nef-ness on nonsingular toric surfaces.

Proposition 1 *Let X be a nonsingular complete toric surface and let D be a T_N-invariant divisor with $|D| \neq \emptyset$. If $|D|$ has no fixed components, then it is free from base points.*

Proof. Since $\Delta(1) = \{\rho_1, \ldots, \rho_s\}$ consists of half-lines from the origin in the plane $N_{\mathbb{R}}$, we may assume that ρ_i and ρ_{i+1} sit next to each other (as usual we consider as $\rho_{s+1} = \rho_0$). Set $\sigma_i = \rho_i + \rho_{i+1} \in \Delta(2)$ for $i = 1, \ldots, s$. Take $D = \sum_i a_i D_i$ with $|D| \neq \emptyset$. We may assume that $a_i \geq 0$ for all i.

First we consider the case that P_D is an integral convex polytope, that is, it is the convex hull of a finite subset of M. Set $H^+(a_i) := \{y \in M_{\mathbb{R}}; \langle y, v_i \rangle \geq -a_i\}$ the half-plane and its boundary line $H(a_i)$. By definition (2) we see that P_D is the intersection of all half-planes $H^+(a_i)$'s. Let u_0 be a vertex of P_D. If $\dim P_D = 2$, then a 1-dimensional face of P_D containing u_0 is contained in some line $H(a_i)$. If $\dim P_D \leq 1$, then P_D itself is contained in some $H(a_i)$. We may set $i = 1$.

Since P_D is the intersection of $H^+(a_i)$'s, we take another line $H(a_j)$ ($j \neq 1$) meeting with $H(a_1)$ at u_0. We may assume that all σ_i with $i = 1, \ldots, j-1$ does not contain $-v_1$. We claim that the line $H(a_i)$ contains u_0 for $i = 2, \ldots, j$.

For $\sigma_i = \rho_i + \rho_{i+1} \in \Delta(2)$, since $\{v_i, v_{i+1}\}$ is a $\mathbb{Z}$-basis of N, there exists $u(\sigma_i) \in M$ satisfying the condition (4). Then we have

$$u_0 \in H^+(a_1) \cap H^+(a_j) \subset u(\sigma_i) + \sigma_i^{\vee}$$

for $i = 1, \ldots, j-1$. If $u(\sigma_1) \neq u_0$, then the half-plane $H^+(a_2 - 1)$ would contain P_D. This implies that D_2 is a fixed component of $|D|$. Then we see that $u(\sigma_1) = u_0$. Considering $v_3, \ldots, v_j$ successively, we see that $u(\sigma_i) = u_0$ for $i = 1, \ldots, j-1$.

When $\dim P_D = 2$, since we can take $H(a_j)$ so that it contains a 1-dimensional face of P_D, we see that the opposite vertex on the edge $H(a_j) \cap P_D$ coincides with $u(\sigma_j)$.

When $\dim P_D \leq 1$, the vector $-v_1$ coincides with some v_k ($j < k$). By the same argument, we see that $u(\sigma_i) = u_0$ for $i = j, \ldots, k-1$. And we see that $u(\sigma_k)$ is also a vertex of P_D. Hence, $\mathcal{O}_X(D)$ is generated by global sections.

Next we assume only that P_D is a rational convex polytope. We can choose a positive integer l so large that lP_D is an integral polytope. Since $lP_D = P_{lD}$, the line bundle $\mathcal{O}_X(lD)$ is generated by global sections, hence it is nef. Then D is nef. On a toric variety, if D is nef, then $\mathcal{O}_X(D)$ is generated by global sections. $\qquad\square$

Remark. If $\dim X \geq 3$, then the same statement of Proposition 1 does not hold. We can easily construct counterexamples, as Professor Payne points out.

2   Adjoint line bundles

Let ω_X be the dualizing sheaf on a toric variety X. If a T_N-invariant Cartier divisor D is ample, then we have (see [18, Proposition 2.24])

$$\Gamma(X, \mathcal{O}_X(D) \otimes \omega_X) \cong \bigoplus_{m \in (\mathrm{Int}(P_D)) \cap M} \mathbb{C}e(m).$$

If we take a resolution $\pi : \tilde{X} \to X$ of singularities by a subdivision of Δ, then $L = \pi^* \mathcal{O}_X(D)$ is nef and big, and we have

$$\Gamma(\tilde{X}, L + K_{\tilde{X}}) \cong \Gamma(X, \mathcal{O}_X(D) \otimes \omega_X).$$

In [20] we show that an ample line bundle L on a nonsingular toric 3-fold X satisfying $h^0(X, L + 2K_X) = 0$ is normally generated. In order to treat more general case, we have to know the adjoint bundle $L + K_X$ with $h^0(L + K_X) \neq 0$.

Lemma 1 *Let X be a nonsingular complete toric variety of dimension three. Suppose that a nef and big line bundle L on X satisfies that $h^0(X, L+K_X) \neq 0$ and that $2L + K_X$ is nef. Let F be the fixed part of $L + K_X$. Then $L + K_X - F$ is nef, F is reduced and for each irreducible component E of the fixed part we have $(E, L_E) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-1)$.*

Proof. By the Mori-Kawamata theory(cf. [8], [13]) if $L + K_X$ is not nef, then we have a contraction morphism $\varphi_1 : X \to Y_1$. Following the same argument of Fujita [4, Theorem 11.8] or [3, Theorem 3], we see that φ_1 contracts an irreducible divisor E_1 to a smooth point and that $(E_1, L_{E_1}) \cong (\mathbb{P}^2, \mathcal{O}(1))$. Moreover, there exists a nef and big line bundle L_1 on Y_1 such that $L + E_1 \cong$

$\varphi_1^* L_1$. Since $K_X = \varphi_1^* K_{Y_1} + 2E_1$, we have $2L + K_X = \varphi_1^*(2L_1 + K_{Y_1})$. Thus $2L_1 + K_{Y_1}$ is nef.

Finally, we have a surjective morphism $\varphi : X \to Y$, which is a composite of blowing-ups of distinct smooth points and there exists a nef and big line bundle $\bar{L}$ on Y such that $\bar{L} + K_Y$ is nef and $L + F \cong \varphi^* \bar{L}$, where $F = \sum_i E_i$ is a sum of exceptional divisors $E_i \cong \mathbb{P}^2$ and $L_{E_i} \cong \mathcal{O}(1)$. $\qquad \square$

Remark. The nef condition for $2L + K_X$ is satisfied if, for example, L is ample with $h^0(L + K_X) \neq 0$ (see [4, Theorems 11.2 and 11.7]). We also have another case satisfying the nef condition. If $-K_X$ is nef and big, then there exists a Gorenstein toric Fano 3-fold Y such that $\pi : X \to Y$ is a crepant resolution of singularities. Thus we have $K_X \cong \pi^* K_Y$. If we take a partial resolution X' of Y with $\phi : X \to X'$ and an ample line bundle L' on X' with $h^0(X', L' + K_{X'}) \neq 0$, then the nef and big line bundle $L = \phi^* L'$ satisfies the nef condition of its adjoint bundle because $2L' + K_{X'}$ is nef from [3, Theorems 1 and 2].

3   A Formula on Minkowski Sums

Let $B := \sum_i D_i$ be the boundary divisor of T_N in X. We assume that B is nef. Then B is nef and big. And there exists a toric 3-fold Y, a surjective morphism $\pi : X \to Y$ and an ample divisor A on Y with $\pi^* A = B$. Since Y has only rational singularities, we have $\pi_* K_X = K_Y$, hence we see that $A = -K_Y$ and Y is a Gorenstein toric Fano 3-fold. We call P_B a *Gorenstein Fano polytope*. From [11] we know that there are 4,319 Gorenstein Fano polytopes of dimension three. In this section we will show a special property of Gorenstein Fano polytopes about Minkowski sums.

Proposition 2 *Let $R \subset M_\mathbb{R}$ be a Gorenstein Fano polytope of dimension three. For any integral convex polytope $Q \subset M_\mathbb{R}$ of dimension three, we have an equality*
$$(R + Q) \cap M + Q \cap M = (R + 2Q) \cap M.$$

Proof. If we decompose as a union $Q = \cup_i Q_i$ of integral convex polytopes Q_i of dimension three such that $Q_i \cap M$ coincides with the set of all vertices of Q_i, then $(\text{Int} Q_i) \cap M = \emptyset$, $R + 2Q = \cup_i (R + 2Q_i)$, and $m \in (R + 2Q) \cap M$ is contained in some $(R + 2Q_i) \cap M$. Thus, for a proof of Proposition it is enough to show the equality
$$(R + Q_i) \cap M + Q_i \cap M = (R + 2Q_i) \cap M \qquad (7)$$
for each Q_i.

Let X be a Gorenstein toric Fano 3-fold with $P_{-K_X} = R$. Unfortunately this Q_i does not always correspond to a nef divisor on X.

In the following we fix i. Let $Y = X(\Delta')$ be the polarized toric 3-fold with the ample line bundle A' corresponding to the polytope Q_i, that is, $P_{A'} = Q_i$. Let $\tilde{\Delta}$ be a nonsingular fan of N which is a refinement of both Δ and Δ'. Let $Z = X(\tilde{\Delta})$ be the nonsingular toric 3-fold defined by the fan $\tilde{\Delta}$, and let $\phi : Z \to X$ and $\psi : Z \to Y$ the morphisms defined by refinements. Then we have two nef divisors $-\phi^* K_X$ and $\psi^* A'$.

Set $L = \mathcal{O}_Z(-\phi^* K_X + \psi^* A')$. For simplicity, we denote $A = \psi^* A'$ on Z. We will show $H^i(Z, L(-iA)) = 0$ for $i \geq 1$.

We have $H^1(Z, L(-A)) = H^1(Z, \phi^* \mathcal{O}_X(-K_X)) = 0$ since $-K_X$ is nef.

From the Serre duality we have $h^3(Z, L(-3A)) = h^0(Z, \mathcal{O}_Z(K_Z + \phi^* K_X + 2A))$. If $\Gamma(\mathcal{O}_Z(K_Z + \phi^* K_X + 2A)) \neq 0$, then we have an injective homomorphism $\mathcal{O}_Z \to \mathcal{O}_Z(K_Z + \phi^* K_X + 2A)$. By tensoring with $\mathcal{O}_Z(-\phi^* K_X)$, we have the injection $\Gamma(Z, \mathcal{O}_Z(-\phi^* K_X)) \to \Gamma(Z, \mathcal{O}_Z(K_Z + 2A))$. This implies the inclusion $R \subset \text{Int}(2Q_i)$, in particular, $R \cap M \subset (\text{Int } 2Q_i) \cap M$. On the other hand, the terminal lemma of White-Frumkin (see, for example, p.48 in [18]) says that there exists an element $m \in M$ and an integer a satisfying

$$a \leq \langle m, y \rangle \leq a + 1 \quad \text{for all } y \in Q_i,$$

since $Q_i \cap M$ coincides with the vertex set of Q_i by definition. Hence the set $(\text{Int } 2Q_i) \cap M$ contained in the plane $\{y \in M_{\mathbb{R}}; \langle m, y \rangle = 2a + 1\}$. This contradicts with $\dim R = 3$. Thus we have $h^3(Z, L(-3A)) = 0$.

From the Serre duality we have $h^2(Z, L(-2A)) = h^1(Z, \mathcal{O}_Z(K_X + A + \phi^* K_X))$. For simplicity we abuse $-K_Z = B$ the sum of all irreducible invariant divisors on Z. Consider the exact sequence

$$0 \to \mathcal{O}_Z(A + \phi^* K_X + K_Z) \to \mathcal{O}_Z(A + \phi^* K_X) \to \mathcal{O}_B((A + \phi^* K_X)_B) \to 0. \quad (8)$$

We note that $H^0(Z, \mathcal{O}_Z(A + \phi^* K_X)) = 0$ since $K_Z = \phi^* K_X + E$ with an effective divisor E and since $H^0(Z, \mathcal{O}_Z(K_Z + A)) = 0$.

We claim that $H^0(B, \mathcal{O}_B((A + \phi^* K_X)_B)) = 0$ and the homomorphism $H^1(Z, \mathcal{O}_Z(A + \phi^* K_X)) \to H^1(B, \mathcal{O}_B((A + \phi^* K_X)_B))$ is injective.

First we note that we have the isomorphism $H^0(Z, \mathcal{O}_Z(A)) \to H^0(B, \mathcal{O}_B(A_B))$ and the surjective homomorphism $H^0(Z, \mathcal{O}_Z(-\phi^* K_X)) \to H^0(B, \mathcal{O}_B(-(\phi^* K_X)_B))$ from the exact sequences

$$0 \to \mathcal{O}_Z(K_Z + A) \to \mathcal{O}_Z(A) \to \mathcal{O}_B(A_B) \to 0$$
$$0 \to \mathcal{O}_Z(K_Z - \phi^* K_X) \to \mathcal{O}_Z(-\phi^* K_X) \to \mathcal{O}_B(-(\phi^* K_X)_B) \to 0$$

and vanishing $H^0(Z, \mathcal{O}_Z(K_Z + A)) = H^1(Z, \mathcal{O}_Z(K_Z + A)) = H^1(Z, \mathcal{O}_Z(K_Z - \phi^* K_X)) = 0$.

If $h^0(B, \mathcal{O}_B((A+\phi^*K_X)_B)) \neq 0$, then we have an injective homomorphism $\mathcal{O}_B((-\phi^*K_X)_B) \to \mathcal{O}_B(A_B)$ from the natural isomorphism $H^0(B, \mathcal{O}_B(A + \phi^*K_X)) \cong \mathrm{Hom}_{\mathcal{O}_B}(\mathcal{O}_B, \mathcal{O}_B((A + \phi^*K_X)_B))$. Thus we have the injective homomorphism $H^0(B, \mathcal{O}_B((-\phi^*K_X)_B)) \to H^0(B, \mathcal{O}_B(A_B))$. By compositing $H^0(Z, \mathcal{O}_Z(-\phi^*K_X)) \to H^0(B, \mathcal{O}_B(-(\phi^*K_X)_B)) \to H^0(B, \mathcal{O}_B(A_B)) \cong H^0(Z, \mathcal{O}_Z(A))$, we have a nontrivial homomorphism $H^0(Z, \mathcal{O}_Z(-\phi^*K_X)) \to H^0(Z, \mathcal{O}_Z(A))$. Since $\mathcal{O}_Z(-\phi^*K_X)$ and $\mathcal{O}_Z(A)$ are generated by their global sections, we have a nontrivial homomorphism $\mathcal{O}_Z(-\phi^*K_X) \to \mathcal{O}_Z(A)$, which defines a nonzero section of $H^0(Z, \mathcal{O}_Z(A + \phi^*K_X))$. This contradicts with $H^0(Z, \mathcal{O}_Z(A + \phi^*K_X)) = 0$. Thus we have $H^0(B, \mathcal{O}_B((A + \phi^*K_X)_B)) = 0$.

Next we take an element $e \in H^1(Z, \mathcal{O}_Z(A + \phi^*K_X))$ such that its image in $H^1(B, \mathcal{O}_B((A + \phi^*K_X)_B))$ is zero. From the natural isomorphism $H^1(Z, \mathcal{O}_Z(A+\phi^*K_X)) \cong \mathrm{Ext}_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{O}_Z(A+\phi^*K_X)) \cong \mathrm{Ext}_{\mathcal{O}_Z}(\mathcal{O}_Z(-\phi^*K_X), \mathcal{O}_Z(A))$, the element e represents an extension

$$0 \to \mathcal{O}_Z(A) \to \mathcal{E} \to \mathcal{O}_Z(-\phi^*K_X) \to 0. \tag{9}$$

The condition on e implies that the extension (9) restricted to B is split, that is, there exists a splitting homomorphism

$$\mu_B : \mathcal{E}_B \to \mathcal{O}_B(A_B).$$

We note that $\mathcal{E}$ is generated by global sections since $\mathcal{O}_Z(A)$ and $\mathcal{O}_Z(-\phi^*K_X)$ are generated by global sections and since $H^1(Z, \mathcal{O}_Z(A))) = 0$. Since the restriction maps $\Gamma(Z, \mathcal{O}_Z(A)) \to \Gamma(B, \mathcal{O}_B(A_B))$ and $\Gamma(Z, \mathcal{O}_Z(-\phi^*K_X)) \to \Gamma(B, \mathcal{O}_B(-(\phi^*K_X)_B))$ are surjective, the restriction map $H^0(Z, \mathcal{E}) \to H^0(B, \mathcal{E}_B)$ is surjective. By compositing

$$H^0(Z, \mathcal{E}) \to H^0(B, \mathcal{E}_B) \xrightarrow{\mu_B} H^0(B, \mathcal{O}_B(A_B)) \xrightarrow{\cong} H^0(Z, \mathcal{O}_Z(A)),$$

we obtain a homomorphism $\mu : \mathcal{E} \to \mathcal{O}_Z(A)$, which gives a splitting of the extension (9). Thus we see that $H^1(Z, \mathcal{O}_Z(A + \phi^*K_X)) \to H^1(B, \mathcal{O}_B((A + \phi^*K_X)_B))$ is injective. From the exact sequence (8) and the clain above, we see the vanishing of $H^1(Z, \mathcal{O}_Z(A + \phi^*K_X + K_Z))$.

From vanishing of $H^i(Z, L(-iA))$ for $i \geq 1$ we can apply [14, Theorem 2] to obtain the surjectivity of the multiplication map

$$\Gamma(Z, \mathcal{O}_Z(A)) \otimes \Gamma(Z, L) \longrightarrow \Gamma(Z, L(A)). \tag{10}$$

This implies the equation $(R + Q_i) \cap M + Q \cap M = (R + 2Q_i) \cap M$. By summing over i, thus, we have $(R + Q) \cap M + Q \cap M = (R + 2Q) \cap M$. This completes the proof of Proposition. $\qquad\square$

Remark. In general, when $\dim Q = \dim R = \operatorname{rank} M = n$ we can prove the equality

$$(R + kQ) \cap M + Q \cap M = (R + (k+1)Q) \cap M \qquad (11)$$

for $k \geq n-1$ by using the toric geometry as above. Proposition 2 implies that when R is a Gorenstein Fano polytope the equality (11) holds for $k = n - 2$. This also gives an answer to the Oda's question[19].

4   Proof of Theorems

Let X be a nonsingular toric variety of dimension three with nef $-K_X$. Let $L = \mathcal{O}_X(D)$ be a nef and big line bundle satisfying the condition in Theorem 1. Then P_D is an integral polytope of dimension three. The assumption $h^0(X, L + K_X) \neq 0$ of Theorem 1 implies that $(\operatorname{Int}(P_D)) \cap M \neq \emptyset$. Let F be the fixed components of $|D + K_X|$ and $A := (D + K_X) - F$. From Lemma 1 we see that $|A|$ is free from base points. Since $\Gamma(X, L + K_X) = \Gamma(X, \mathcal{O}_X(A))$, we see that P_A coincides with the convex full of $(\operatorname{Int}(P_D)) \cap M$. We note that if $-K_X = B$ is nef, then $D - F = A + B$ is also nef.

First, we will prove the normal generation of $\mathcal{O}_X(A + B)$ for any nef A and $B = -K_X$.

Proposition 3 *Let X be a nonsingular toric variety of dimension three with nef anti-canonical divisor and let A be a nef divisor on X. Then the line bundle $\mathcal{O}_X(-K_X + A)$ is normally generated.*

Before treating nef divisors on 3-folds, we need to treat the more about nef divisors on toric surfaces. For a proof of the following lemma we use the result of Haase, Nill, Paffenholz and Santos[6], or Kondo and Ogata[10], which is a generalization of the result obtained by Fakhruddin[2] to the case of singular toric surfaces.

Lemma 2 *Let A and B be nef divisors on a nonsingular complete toric surface Y. Then the multiplication map of global sections*

$$\Gamma(Y, \mathcal{O}_Y(A)) \otimes \Gamma(Y, \mathcal{O}_Y(A + B)) \longrightarrow \Gamma(Y, \mathcal{O}_Y(2A + B))$$

is surjective.

Proof. Since $\dim Y = 2$, in this proof we set $M \cong \mathbb{Z}^2$ and $P_A, P_B \subset M_{\mathbb{R}} \cong \mathbb{R}^2$. We will show the equality

$$P_A \cap M + (P_A + P_B) \cap M = (2P_A + P_B) \cap M. \qquad (12)$$

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When $\dim P_{A+B} = 1$ we see that $\dim P_A = \dim P_B = 1$, hence, the equality (12) trivially holds.

When $\dim P_{A+B} = 2$, take the normal fan Δ of P_{A+B}. Then the toric surface $Z = X(\Delta)$ has the ample line bundle L with $P_L = P_{A+B}$ and Y is a resolution of singularities of Z. By definition A and B are also nef divisors on Z. From [6, Theorem 1.1] or [10, Theorem 1], the equality (12) holds. $\square$

We return to the case that $B = -K_X$ and $A = D + K_X - F$ in dimension three.

Proof of Proposition 3. Set $L = \mathcal{O}_X(A + B)$. Since $\mathcal{O}_X(A) = L + K_X$, we have an exact sequence

$$0 \to \mathcal{O}_X(A) \to L \to L_B \to 0. \tag{13}$$

Since A is nef, we have $H^i(X, \mathcal{O}_X(A)) = 0$ for $i \geq 1$. Thus the sequence of the global sections of (13) is exact.

Take the tensor product with $\Gamma(X, \mathcal{O}_X(A))$. When $\dim P_A \leq 2$, we see that $\mathcal{O}_X(A)$ is normally generated (see (1)).

On the other hand, $\Gamma(B, (2L + K_X)_B)$ has a basis $\{e(m); m \in (\partial(2P_A + P_B)) \cap M\}$ as vector spaces. One $e(m)$ is contained in $\Gamma(D_i, (2L + K_X)_{D_i})$ for some D_i. Since the restriction map $\Gamma(X, G) \to \Gamma(D_i, G_{D_i})$ is surjective for any nef line bundle G on a toric variety X, from Lemma 2 we see that the multiplication map

$$\Gamma(B, L_B) \otimes \Gamma(B, (L + K_X)_B) \longrightarrow \Gamma(B, (2L + K_X)_B)$$

is surjective. Thus we obtain the surjectivity of $\Gamma(L) \otimes \Gamma(L + K_X) \to \Gamma(2L + K_X)$ when $\dim P_A \leq 2$. From Proposition 2, we see that this multiplication map is also surjective when $\dim P_A = 3$.

By tracing the same argument after changing A with $L = A + B$, we obtain a proof of the normal generation of $\mathcal{O}_X(A + B)$. $\square$

Proof of Theorem 1. Let L be a nef and big line bundle on X satisfying the condition that $2L + K_X$ is nef and $h^0(X, L + K_X) \neq 0$.

If $L + K_X$ has no fixed components, then we see the normal generation of L from Proposition 3. Let F be the fixed components of $L + K_X$. By the condition that $2L + K_X$ is nef, we see from Lemma 1 that $F = \sum_i E_i$, $E_i \cong \mathbb{P}^2$ and E_i's are disjoint. And we have $L_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $L(-F)_{E_i} \cong \mathcal{O}_{\mathbb{P}^2}(2)$.

Consider the exact sequence

$$0 \to L(-F) \to L \to L_F \to 0. \tag{14}$$

Since $L(-F)$ is nef, we have $H^1(X, L(-F)) = 0$. Thus the sequence of global sections of (14) is exact. Taking the tensor product with $\Gamma(X, L(-F))$, we

see the surjectivity of the map

$$\Gamma(X, L(-F)) \otimes \Gamma(X, L) \longrightarrow \Gamma(X, 2L(-F))$$

since $L(-F)$ is normally generated from Proposition 3. By changing the role of $\Gamma(X, L(-F))$ with $\Gamma(X, L)$ we see the normal generation of L. $\qquad\square$

Proof of Theorem 2. Let L be an ample line bundle on a nonsingular toric 3-fold X with nef $-K_X$. If L satisfies $h^0(L + K_X) = 0$, then it is normally generated from [20, Proposition 2].

If $h^0(L + K_X) \neq 0$, then $2L + K_X$ is nef from[4, Theorems 11.2 and 11.7], hence, this ample line bundle L satisfies the condition of Theorem 1. $\qquad\square$

If the anti-canonical divisor $-K_X$ of a nonsingular toric variety X is nef, then it is nef and big, hence, there exists a polarized toric variety (Y, A) and a surjective morphism $\pi : X \to Y$ such that $-K_X \cong \pi^* A$. Since Y has only rational singularity, we see that $A = -K_Y$ and Y is Gorenstein.

On the other hand, let Y be a Gorenstein toric Fano 3-fold. Then we have a resolution of singularities $\pi : X \to Y$ with $K_X \cong \pi^* K_Y$. If an ample line bundle L on Y satisfies $h^0(L + K_Y) \neq 0$, then $2L + K_Y$ is nef from [3, Theorems 1 and 2]. Thus we can apply Theorem 1 to a nef and big line bundle $\pi^* L$ on X. Since $\Gamma(X, \pi^* L^{\otimes l}) \cong \Gamma(Y, L^{\otimes l})$, we obtain a proof of Theorem 3.

In Theorem 1 or 3 we cannot remove the condition $h^0(X, L + K_X) \neq 0$. We give an example of (X, L) such that $-K_X$ is nef but L is not normally generated and $h^0(X, L + K_X) = 0$.

Let $M = \mathbb{Z}^3$ and $P := \mathrm{Conv}\{0, (1, 0, 0), (0, 1, 0), (1, 1, 2)\}$ in $M_{\mathbb{R}}$. Then there exists the polarized toric 3-fold $(Y, \mathcal{O}_Y(D))$ with $P_D = P$. This Y is Gorenstein toric Fano with $-K_Y = 2D$. Since P does not contain lattice points of the form $(a, b, 1)$, we can easily see that D is not very ample. We can make a toric resolution $\pi : X \to Y$ of singularities with $K_X = \pi^* K_Y$. Then $-K_X$ is nef (and big) and $L := \pi^* \mathcal{O}_Y(D)$ is nef and big, and $h^0(X, L + K_X) = 0$.

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